*Journal of Statistical Physics, Vol. 56, Nos. 5/6, 1989*

# Perturbation Expansion for a One-Dimensional Anderson Model with Off-Diagonal Disorder

**Anton Bovier**[1]

*Received March 7, 1989; revision received March 28, 1989*

The weak disorder expansion for a random Schrödinger equation with off-diagonal disorder in one dimension is studied. The invariant measure, the density of states, and the Lyapunov exponent are computed. The most interesting feature in this model appears at the band center, where the differentiated density of states diverges, while the Lyapunov exponent vanishes. The invariant measure approaches an atomic measure concentrated on zero and infinity. The results extend previous work of Markos to all orders of perturbation theory.

**KEY WORDS:** Random Schrödinger operators; density of states; Lyapunov exponent; invariant measure; perturbation expansion.

## 1. INTRODUCTION

In a recent paper, Bovier and Klein[(2)] investigated the weak disorder expansion for the one-dimensional random Schrödinger operator

$$H = -\Delta + \lambda V \tag{1.1}$$

where $\Delta$ is the off-diagonal part of the finite-difference Laplacian on $L^2(\mathbb{Z})$ and $V$ is a diagonal matrix whose diagonal elements are independent identically distributed (i.i.d.) random variables. Assuming the existence of moments of all orders for the common distribution of these random variables, we showed that the invariant measure and hence the Lyapunov exponent and the density of states permit a perturbative expansion in powers of $\lambda$ with finite coefficients to all orders. Moreover, we showed that at energies $E = 2\cos(\pi(p/q))$, where $p$ and $q$ are relatively prime integers, the ($q$-2)th coefficients in this expansion are discontinuous as functions of

[1] Physikalisches Institut, Universität Bonn, Universität Bonn, D-5300 Bonn, West Germany.



0022-4715/89/0900-0645$06.00/0 

$E$; this phenomenon of anomalies had previously been observed at $E=0$ and at $E=\pm 1$ by several authors (refs. 4 and 7; see also ref. 8).

In (1.1) the disorder is manifest in a random potential. In other physical situations, the main source of randomness may be that the tunneling probabilities between potential wells depend on the atoms occupying the sites. This situation may be modeled by a Hamiltonian

$$H = -\Delta + \lambda J \tag{1.2}$$

where

$$J_{ij} = \begin{cases} v_j & \text{if } i-j=1 \\ v_i & \text{if } j-i=1 \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

As before, the $v_i$ are i.i.d. random variables, the distribution of which we assume to have moments of all orders. We also put

$$\mathbf{E} v_i = 0, \qquad \mathbf{E} v_i^2 = 1$$

(We denote by $\mathbf{E}$ the expectation with respect to the distribution $d\mu$ of the $v_i$.)

This model has been studied by Theodorou and Cohen[13] and Roman and Wiecko,[11] and more recently by Markos[8,9] and Campanino and Perez.[3] A closely related problem of a disordered chain of coupled harmonic oscillators was already studied by Dyson.[5] Markos[9] investigated the behavior of the density of states and the Lyapunov exponent near the band center ($E=0$) in perturbation theory up to second order in $\lambda$, using essentially the techniques of Derrida and Gardner[4] developed in the study of the band center anomalies in the model (1.1). Markos finds:

(i) The Lyapunov exponent $\gamma(E)$ vanishes like $1/\ln E$.

(ii) The differentiated density of states $n(E)$ diverges like $1/(E \ln^3 E)$.

This is in agreement with the formulas of ref. 13 based on computations of Dyson in a particular potential distribution,[5] with the numerical results of ref. 11, and also with the rigorous bounds of ref. 3 (which, however, do not reproduce the exact nature of the singular behavior at $E=0$).

This interesting phenomenon deserves further careful investigation. In the present paper I extend the results of Markos in several directions:

(i) I construct the perturbation expansion for the invariant measure to all orders and show that the above-mentioned behavior holds to all orders of perturbation theory.

(ii) I show that anomalies of the same type as in the random potential model (1.1) appear in $(q-2)$th order of perturbation theory at energies $E = 2\cos(\pi(p/q))$.

(iii) I study a model with both diagonal and off-diagonal disorder, and show that the presence of an arbitrarily small amount of diagonal disorder removes the singular behavior at $E = 0$.

Before turning to the more specific computations, I add some comments on the physical interpretation of this phenomenon. Markos[9] concludes that in spite of the vanishing of the Lyapunov exponent at $E = 0$, the electrons there remain exponentially localized. He bases his arguments on the divergence of what he calls "moments of the Lyapunov exponent." He seems to conclude that the Lyapunov exponent takes on different values in different realizations of the system, the value zero having zero probability.

Theodorou and Cohen[13] claim, on the contrary, that extended states exist at the band center, since the localization length diverges, i.e., $\gamma(0) = 0$.

Both conclusions seem to me somewhat premature. The argument of Markos is based in fact on a misconception regarding the nature of the quantities he computes. It seems worthwhile to clarify this point. I begin by recalling the definition of the Lyapunov exponent and the invariant (or stationary) measure. (For an extended treatment, see, e.g., the book by Bougerol and Lacroix.[1])

Let $\psi_E(n)$ denote a solution of the Schrödinger equation

$$(H\psi_E)(n) = E\psi_E(n) \tag{1.4}$$

with initial conditions $\psi_E(0)$, $\psi_E(1)$. The Lyapunov exponent $\gamma(E)$ is defined as

$$\gamma(E) = \lim_{n \to \infty} \frac{1}{n} \ln \left\| \prod_{k=1}^{n} P_k \right\| \tag{1.5}$$

where $P_k$ is the (random) transfer matrix associated with Eq. (1.4), i.e.,

$$\begin{pmatrix} \psi_E(n+1) \\ \psi_E(n) \end{pmatrix} = P_n \begin{pmatrix} \psi_E(n) \\ \psi_E(n-1) \end{pmatrix} \tag{1.6}$$

The subadditive ergodic theorem (see, e.g., ref. 10) implies that the limit in (1.5) exists and is independent of the realization of the disorder, for almost all realizations. In other words, $\gamma(E)$ is self-averaging,

$$\mathbf{E}\gamma(E) = \gamma(E) \tag{1.7}$$

If the matrices $P_n$ satisfy some irreducibility criteria (see, e.g., ref. 1), the Lyapunov exponent can also be expressed as

$$\begin{aligned}\gamma(E) &= \lim_{n\to\infty} \frac{1}{n} \ln \left|\frac{\psi_E(n)}{\psi_E(0)}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln \left|\frac{\psi_E(k+1)}{\psi_E(k)}\right| \end{aligned} \tag{1.8}$$

As I show in more detail later, the ratios $x_n \equiv \psi_E(n+1)/\psi_E(n)$ satisfy a random recursion relation of the form $x_n = \tau_{v_n}(x_{n-1})$ and in fact form a Markov chain. If this chain is ergodic [as is the case for $\lambda > 0$ and $E \neq 0$ in the model (1.2)], the limit in (1.8) exists and is independent of the initial data. It then follows from (1.7) that for any invariant measure for our chain, i.e., a measure $dv(x)$ such that for all measurable functions $f(x)$

$$\int dv(x)\, f(x) = \mathbf{E} \int dv(x)\, f(\tau_v(x)) \tag{1.9}$$

$\gamma(E)$ can be expressed in terms of an expectation with respect to this measure

$$\gamma(E) = \int \ln |x|\, dv_{\lambda,E}(x) \tag{1.10}$$

The quantities Markos introduces as "moments of the Lyapunov exponent" are essentially the logarithmic moments of this invariant measure, i.e.,

$$\langle \gamma(E)^n \rangle \sim \int \ln^n |x|\, dv(x)$$

and are not to be confused with $\mathbf{E}\gamma(E)^n$. In fact,

$$\int \ln^n |x|\, dv(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln^n \left|\frac{\psi_E(n+1)}{\psi_E(n)}\right|$$

and their divergence for $n > 1$ as $E \to 0$ seem to indicate that a typical solution of the initial value problem has large oscillations everywhere that, since $\gamma(E) \to 0$, accumulate to less than exponentially fast growth. Thus, we must conclude that the Lyapunov exponent does vanish for almost every realization of the disorder.

While this leaves the claim of ref. 13 as a possible answer, I do not see how this proves the existence of extended states. To answer this question, more detailed information on the growth properties of the solutions is required. By going one step beyond the computations of ref. 13, I will show that for $E=0$ the initial value problem has two solutions that behave for large $n$ like

$$\psi_0^{\pm}(n) \sim \exp(\pm K n^{1/2}) \tag{1.11}$$

where $K$ is a random variable with mean zero and variance of order $\lambda$. This leads to the conjecture that the wave function is localized even in the center of the band, but decaying away from the localization region only like an exponential of the square root of the distance. I have not been able, however, to prove this conjecture. This point certainly deserves further clarification.

The remainder of this paper is organized as follows. In Section 2, I derive the equation for the invariant measure and some formulas relating it to the Lyapunov exponent and the density of states. I also derive the equations governing the perturbation expansion, both at special and nonspecial energies. In Section 3, I consider nonspecial energies and show that the perturbation expansion can be constructed to all orders. In Section 4, I do the same at the special energies $E=2\cos(\pi(p/q))$, with the exception of $E=0$. I also show the appearence of anomalies in order $q-2$. Both of these sections are largely adaptations of results from ref. 2. In Section 5, I study the neighborhood of the band center, where I extend the results of Markos to all orders of perturbation theory. Finally, in Section 6, I discuss the situation in models with both diagonal and off-diagonal disorder and present some discussions and conclusions.

## 2. EQUATIONS FOR THE INVARIANT MEASURE

The Schrödinger equation associated with the Hamiltonian (1.2) reads

$$\psi_E(n+1)+\psi_E(n-1)+\lambda v_{n+1}\psi_E(n+1)+\lambda v_n\psi_E(n-1)-E\psi_E(n)=0 \tag{2.1}$$

Following ref. 8, I introduce the ratios

$$x_n \equiv \frac{\psi_E(n)}{\beta_n \psi_E(n-1)} \in \dot{\mathbf{R}} \tag{2.2}$$

where $\beta_n \equiv 1+\lambda v_n$, and $\dot{\mathbf{R}}$ denotes the compactified real line.

The $x_n$ satisfy the recursion relation

$$x_{n+1} = \frac{E - x_n^{-1}}{\beta_{n+1}^2} \tag{2.3}$$

The $(x_n, \beta_{n+1})$ form a Markov process. The complex Lyapunov exponent is easily expressed in terms of these variables. We have

$$\begin{aligned}\tilde{\gamma}(E) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln(\beta_n x_n) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln x_n + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln \beta_n \end{aligned} \tag{2.4}$$

The process defined through (2.3) is ergodic for $\lambda > 0$ unless $E = 0$! [And also if $\lambda = 0$, but $E \neq 2\cos(\pi(\pi(p/q))$. If $E = 0$, there exists an invariant set, $\{0, \infty\} \otimes \{1 + \lambda\, \mathrm{supp}(\mu)\}$. In this situation, Furstenberg's theorem[1,6] does not apply, and in particular we are not guaranteed the uniqueness of the invariant measure and the positivity of the Lyapunov exponent. For all $E \neq 0$, on the other hand, Furstenberg's theorem asserts that there is a unique invariant measure $d\nu_{\lambda,E}(x)$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln x_n = \int_{\mathbf{R}} d\nu_{\lambda,E}(x) \ln x \tag{2.5}$$

Therefore,

$$\tilde{\gamma}(E) = \int_{\mathbf{R}} d\nu_{\lambda,E}(x) \ln x + \mathbf{E} \ln \beta \tag{2.6}$$

and

$$\gamma(E) = \Re\tilde{\gamma}(E) = \int_{-\infty}^{\infty} d\nu_{\lambda,E}(x) \ln |x| + \mathbf{E} \ln \beta \tag{2.7}$$

and[2]

$$N(E) = \Im\tilde{\gamma}(E) = \int_{-\infty}^{0} d\nu_{\lambda,E}(x) \tag{2.8}$$

Our problem is now to find the invariant measure $d\nu_{\lambda,E}(x)$. It satisfies the equation

$$\int d\nu_{\lambda,E}(x)\, f(x) = \mathbf{E} \int d\nu_{\lambda,E}(x)\, f\left(\frac{E - x^{-1}}{\beta^2}\right) \tag{2.9}$$

[2] Relating the imaginary part of $\tilde{\gamma}(E)$ to the density of states employs the Sturm oscillation theorem. See, e.g., ref. 12.

for all measurable functions. We may try to seek an invariant measure that has a density, i.e., for which $d\nu_{\lambda,E}(x)=\phi_{\lambda,E}(x)\,dx$. Equation (2.9) then implies that $\phi_{\lambda,E}(x)$ must satisfy

$$\phi_{\lambda,E}(x)=\mathbf{E}\left[\frac{\beta^2}{(E-\beta^2x)^2}\,\phi_{\lambda,E}\left(\frac{1}{E-\beta^2x}\right)\right] \tag{2.10}$$

which can be conveniently written in the form

$$\phi_{\lambda,E}(x)=\mathbf{E}\left[\exp\left(2\ln\beta\,\frac{d}{dx}\,x\right)\right]T_E\phi_{\lambda,E}(x) \tag{2.11}$$

where $T_E$ is defined as[2]

$$(T_Ef)(x)=\frac{1}{(E-x)^2}\,f\left(\frac{1}{E-x}\right) \tag{2.12}$$

A solution of Eq. (2.10) is of course sought among the nonnegative functions in $L^1(\mathbf{R}, dx)$. We recall from ref. 2 that for $E=2\cos(\pi(p/q))$, with $p$, $q$ relatively prime integers,

$$T_E^q=id$$

and the spectrum of $T_E$ consists of the qth roots of unity. I refer to these energies as "special energies." For all other energies with $-2<E<2$, $T_E$ has a simple eigenvalue one, and the rest of the spectrum is a continuous spectrum on the unit circle.

Equation (2.11) is a convenient starting point for a perturbation theory in $\lambda$. We expand (formally) $\phi_{\lambda,E}$ in powers of $\lambda$,

$$\phi_{\lambda,E}(x)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,\phi_E^{(n)}(x) \tag{2.13}$$

Equation (2.12) then implies that the coefficients $\phi_E^{(n)}(x)$ satisfy the infinite system of equations

$$\begin{aligned}(1-T_E)\,\phi_E^{(n)}(x)&=\sum_{k=2}^{n}\binom{n}{k}\left[\frac{\partial^k}{\partial\lambda^k}\exp\left(2\ln\beta\,\frac{d}{dx}\,x\right)\right]_{\lambda=0}T_E\phi_E^{(n-k)}(x)\\&=\sum_{k=2}^{n}\binom{n}{k}\mathbf{E}v^k\prod_{r=0}^{k-1}\left(2\frac{d}{dx}x-r\right)T_E\phi_E^{(n-k)}(x)\end{aligned} \tag{2.14}$$

One would like to solve the system of equations (2.14) recursively. This succeeds, as I show briefly in the next section, if and only if $E$ is a nonspecial energy, since in this case the kernel of $(1 - T_E)$ is one dimensional. For energies in the neighborhoods of the special energies $E = 2\cos(\pi\, p/q)$ we must derive the equations for the perturbation expansion from the $q$-fold iterated equation, i.e., from

$$B^q_{\lambda, E}\phi_{\lambda, E} = \phi_{\lambda, E} \tag{2.15}$$

For $E = 2\cos(\pi\, p/q) + \lambda^2\varepsilon$, we then obtain the equations

$$\binom{n}{2} A_{0,\varepsilon}\phi_E^{(n-2)}(x) = -\sum_{k=3}^{n} \binom{n}{k}\left[\frac{\partial^k}{\partial\lambda^k} B^q_{\lambda, E_0+\lambda^2\varepsilon}\right]_{\lambda=0} \phi_E^{(n-k)}(x) \tag{2.16}$$

with

$$\begin{aligned} A_{0,\varepsilon} &= \left[\frac{\partial^2}{\partial\lambda^2} B^q_{\lambda, E_0+\lambda^2\varepsilon}\right]_{\lambda=0} \\ &= \sum_{k=0}^{q-1} T^k_{E_0}\left[4\left(\frac{d}{dx}x\right)^2 - 2\frac{d}{dx}x + \varepsilon\frac{d}{dx}\right] T^{q-k}_{E_0} \end{aligned} \tag{2.17}$$

I will show that the operator $A_{0,\varepsilon}$ has a one-dimensional kernel and that Eq. (2.16) can be solved recursively, except when $E_0 = 0$, and $\varepsilon = 0$. In this latter case, the lowest order equation $A_{0,\varepsilon}\phi_E^{(0)}(x) = 0$ does not have a normalizable solution.

## 3. THE NONSPECIAL ENERGIES

For $E = 2\cos\pi\alpha$ with $\alpha$ irrational, the unique solution of (2.14) can be constructed as in the random potential problem.[2] I briefly indicate how this is done, but refer to ref. 2 for details. It turns out that a solution can be constructed in the Hilbert spaces $\mathscr{H}_E \equiv L^2(\mathbf{R}, (x^2 - Ex + 1)\, dx) \subset L^1(\mathbf{R}, dx)$. We introduce the complete orthogonal set $\{P^E_m\}^{\infty}_{m=-\infty}$ where

$$P^E_m(x) = \frac{1}{x^2 - Ex + 1}\, e^{2im \cot^{-1}(x - \cot\pi\alpha)} \tag{3.1}$$

and expand

$$\phi_E^{(m)}(x) = \sum_{m=-\infty}^{\infty} \hat{\phi}_E^{(n)}(m)\, P^E_m(x) \tag{3.2}$$

A simple computation shows that the $\hat{\phi}_E^{(n)}(m)$ satisfy the system of equations

$$\begin{aligned} &(1 - e^{-i2\pi m\alpha})\,\hat{\phi}_E^{(n)}(m) \\ &\quad = \sum_{k=2}^{n} \binom{n}{k} \mathbf{E} v^k \sum_{l=-\infty}^{\infty} \prod_{r=0}^{k-1} (M - r)_{ml}\, e^{-i2\pi l\alpha} \hat{\phi}_E^{(n-k)}(l) \end{aligned} \tag{3.3}$$

where $M$ is a matrix with elements

$$M_{ml} = im[\delta_{m,l-1}(\cot \pi\alpha - 1) + \delta_{m,l+1}(\cot \pi\alpha + 1) - \delta_{m,l} 2 \cot \pi\alpha] \quad (3.4)$$

Equation (3.3) is completely analogous to the corresponding equation (3.7) in ref. 2. In particular, the matrix $M$ is tridiagonal, like the matrix $D$ in ref. 2. The same arguments as in Chapter III of ref. 2 show therefore that we have the following result.

**Lemma 1.** For $E = 2\cos \pi\alpha$ with $\alpha$ irrational, (3.3) has a unique normalized set of solutions. Moreover,

$$\hat{\phi}_E^{(0)}(m) = \delta_{m,0}$$

$$\hat{\phi}_E^{(n)}(0) = \delta_{n,0}$$

$$\hat{\phi}_E^{(n)}(m) = 0 \qquad \text{if} \quad |m| > n$$

The $\phi_E^{(n)}(x)$ may be continued to rational points $\alpha = p/q$ if $n < q$. If $n = q$, the coefficient $\hat{\phi}_E^{(q)}(q)$ diverges at these energies. Moreover, from this divergence we may conclude that for some $n \leqslant q - 2$, the continuation of the $\phi_E^{(n)}(x)$ from the irrational to the rational point cannot give the correct solution, and that thus the correct $\phi_E^{(n)}(x)$ are discontinuous at these special energies.

It should be noted that, for generic $\alpha$, the solutions of (3.3) develop a small-divisor problem, as $(1 - e^{-2\pi m\alpha})$ will become arbitrarily small. For sufficiently irrational $\alpha$, this problem can be avoided, and one might hope that in these cases the perturbation expansion is Borel summable. (It is fairly easy to see that the coefficients have only factorial growth.) However, I have not been able to prove such a result. For $\alpha$'s closer to rationals, one should use the expansion developed in the next section.

## 4. ANOMALIES AT $E_0 \neq 0$

I turn now to the special energies. Only the case with $E = 0$ will be substantially different from the random potential case, and its treatment is deferred to the next section. The other energies can be treated in complete analogy to the random potential model. I will only give a brief outline and the major results here, while referring to Chapter four of ref. 2 for details.

I recall from ref. 2 the following key lemma:

**Lemma 2.** Let $E_0 = 2\cos(\pi\, p/q)$, with $p$ and $q$ relatively prime integers. Let $D^{(m-1)}$ be a differential expression in $d/dx$ with analytic coefficients of degree $m - 1$. If $D^{m-1}$ satisfies

$$x^2 T_{E_0}^{-1} D^{(m-1)} T_{E_0} = D^{(m-1)}$$

and if $m < q$, then there exist constants $c_k$ such that

$$D^{(m-1)} = r_{E_0}(x) \sum_{k=1}^{m} c_k \left[ \frac{d}{dx} r_{E_0}(x) \right]^{k-1}$$

where $r_{E_0}(x) = x^2 - E_0 x + 1$.

We may apply this lemma immediately to the operator $A_{0,\varepsilon}$. It is obviously of the form $A_{0,\varepsilon} = (d/dx)\, D^{(1)}$. Since

$$T_{E_0}^{-1} A_{0,\varepsilon} T_{E_0} = A_{0,\varepsilon}, \qquad T_{E_0}^{-1} \frac{d}{dx} T_{E_0} = \frac{d}{dx} x^2$$

we see that $D^{(1)}$ satisfies the assumption of the lemma, and therefore, if $q > 2$,

$$A_{0,\varepsilon} = c_1 \frac{d}{dx} r_{E_0}(x) \frac{d}{dx} r_{E_0}(x) + c_2(\varepsilon - 2) \frac{d}{dx} r_{E_0}(x) \tag{4.1}$$

with some constants $c_1$, $c_2$. It follows that $\phi_E^{(0)}(x) = 1/r_{E_0}(x)$ (up to the proper normalization factor) and that the system of equations (2.17) can be solved recursively in the space $\mathscr{H}_{E_0}$. Moreover, in exactly the same way as in ref. 2, one shows that for $n \leqslant q-3$ the solutions so obtained coincide with the continuations of the solutions obtained in the previous section for nonspecial energies to $E_0$. The $(q-2)$th coefficient is, however, discontinuous at $E_0$.

## 5. THE BAND CENTER

I now come to the most interesting feature of this model, namely the behavior of the invariant measure near the band center. The operator $A_{0,\varepsilon}$ is in this case not determined by Lemma 2, but one can easily compute it explicitly. One finds

$$A_{0,\varepsilon} = 8 \left( \frac{d}{dx} x \right)^2 + 2\varepsilon \frac{d}{dx} (1 + x^2) \tag{5.1}$$

For $\varepsilon = 0$, we see that the lowest order equation

$$A_{0,0} \phi_E^{(0)}(x) = 0$$

has the two solutions $1/|x|$ and $\ln|x|/|x|$, both of which are not integrable. (The operator $A_{0,0}$ has in fact purely continuous spectrum.) This is of course not too surprising in view of the fact that there is at least one invariant measure that is not absolutely continuous, namely

$$d\nu(x)=\frac{1}{2}\left[\delta(x)+\frac{1}{x^2}\delta\left(\frac{1}{x}\right)\right]dx \tag{5.2}$$

which is concentrated on zero and infinity. This also happens to be the only invariant measure, since the complement of the set $\{0, \infty\}$ is transient.

For $\varepsilon \neq 0$, this problem does not yet appear, and we will solve our equations for $\varepsilon$ finite and extract the singular behavior as $\varepsilon \to 0$. The general solution of $A_{0,\varepsilon}\phi(x)=0$ is easily found as

$$\begin{aligned}\phi(x)&=c_1\frac{1}{|x|}e^{(\varepsilon/x-\varepsilon x)/4}\\&\quad+c_2\frac{1}{x}e^{(\varepsilon/x-\varepsilon x)/4}\int_{a(x)}^{x}\frac{1}{y}e^{(\varepsilon y-\varepsilon/y)/4}\,dy\end{aligned} \tag{5.3}$$

The integration constant $a(x)$ is choosen as

$$a(x)=\begin{cases}-\infty & \text{for } x<0\\ 0 & \text{for } x>0\end{cases} \tag{5.4}$$

Obviously, this function is normalizable only if $c_1=0$. I will compute the norm of the second term shortly. I put thus

$$\phi_\varepsilon^{(0)}(x)=\frac{1}{x}e^{(\varepsilon/x-\varepsilon x)/4}\int_{a(x)}^{x}\frac{1}{y}e^{(\varepsilon y-\varepsilon/y)/4}\,dy \tag{5.5}$$

Notice that zero is a singular point for our differential operator, and with our choice of $a(x)$ we have matched solutions for $x<0$ and $x>0$ in such a way that

$$T_0\phi_\varepsilon^{(0)}(x)=\phi_\varepsilon^{(0)}(x) \tag{5.6}$$

as it should be. It will turn out to be useful to introduce new variables in the following way. For $x>0$ we set $t=\ln x$ and define

$$g_\varepsilon^{(0)}(t)=x\phi_\varepsilon^{(0)}(x)=e^{-(\varepsilon/2)\sinh t}\int_{-\infty}^{t}ds\,e^{(\varepsilon/2)\sinh s} \tag{5.7}$$

For $x<0$ we let $t=\ln|x|$, and put $f_\varepsilon^{(0)}(t)=|x|\,\phi_\varepsilon^{(0)}(x)$. Notice that by (5.6) for $x<0$,

$$f_\varepsilon^{(0)}(t)=g_\varepsilon^{(0)}(-t) \tag{5.8}$$

In the same way we introduce $g_\varepsilon^{(n)}(t)$ and $f_\varepsilon^{(n)}(t)$.

As a first exercise we will use these variables to compute the norm of $\phi_\varepsilon^{(0)}(x)$. We have

$$\begin{aligned}\|\phi_\varepsilon^{(0)}(x)\|_1 &= \int_{-\infty}^{0} dx\, \phi_\varepsilon^{(0)}(x)+\int_0^\infty dx\,\phi_\varepsilon^{(0)}(x)\\ &= 2\int_{-\infty}^{\infty} dt\, e^{-(\varepsilon/2)\sinh t}\int_{-\infty}^{t} ds\, e^{(\varepsilon/2)\sinh s}\end{aligned} \tag{5.9}$$

This integral can in fact be computed exactly and yields[9]

$$\|\phi_\varepsilon^{(0)}(x)\|_1=\tfrac{1}{2}\pi^2[J_0^2(\varepsilon/2)+N_0^2(\varepsilon/2)]\sim[\ln\varepsilon]^2 \tag{5.10}$$

In view of later necessities, it is, however, more instructive to estimate the integral in (5.9) in the following way. We change variables once again to $T=\varepsilon\sinh t$, $S=\varepsilon\sinh s$, so that our integral takes the form

$$\begin{aligned}\|\phi_\varepsilon^{(0)}(x)\|_1 &= 2\int_{-\infty}^{\infty}\frac{dT\,e^{-T/2}}{(\varepsilon^2+T^2)^{1/2}}\int_{-\infty}^{T}\frac{dS\,e^{+S/2}}{(\varepsilon^2+S^2)^{1/2}}\\ &= 2\int_{-\infty}^{\infty}\frac{dT}{(\varepsilon^2+T^2)^{1/2}}\int_0^\infty\frac{d\tilde T\,e^{-\tilde T/2}}{[\varepsilon^2+(T-\tilde T)^2]^{1/2}}\end{aligned} \tag{5.11}$$

The first $T$-integral converges since the denominator behaves like $T^2$ and picks up two contributions of order $\ln\varepsilon$ near $T=0$ and near $T=\tilde T$. Thus,

$$\|\phi_\varepsilon^{(0)}(x)\|_1\approx 2\,|\ln|\,\varepsilon\|\int_0^\infty\frac{d\tilde T\,2e^{-\tilde T/2}}{(\varepsilon^2+\tilde T^2)^{1/2}}$$

The convergence of the $\tilde T$ integral is assured by the exponential damping term and its main contribution comes from the neighborhood of zero, where it picks up another factor of $\ln|\varepsilon|$. This coincides with the result of the exact computation.

One can expect our perturbation expansion to describe correctly the singular behavior of the full solution only if the nature of the singularity in $\varepsilon$ is the same in all orders of perturbation theory. In particular, the norms of all the $\phi_\varepsilon^{(n)}(x)$ should diverge with the same speed. This question has also been raised by Markos. The following theorem gives a complete answer to this question.

**Theorem 1.** Let $E=\lambda^2\varepsilon$. Then, for all $\varepsilon\neq 0$, the system of equations (2.16) has a unique [up to a constant multiple of $\phi_\varepsilon^{(0)}(x)$, which is to be determined by the normalization condition] set of finite solutions $\phi_\varepsilon^{(n)}(x)$ in $L^1$. Moreover, for all $n$,

$$\|\phi_\varepsilon^{(n)}(x)\|_1 \leqslant c_n \,|\ln|\,\varepsilon\|^2 \tag{5.12}$$

with $c_n$ constants independent of $\varepsilon$.

*Proof.* The existence and uniqueness of a set of finite solutions to all orders of (2.16) can be shown very easily along the lines of the analogous proofs in ref. 2 or the previous section of this article. All it really requires is the fact that the kernel of $A_{0,\varepsilon}$ is one dimensional. What is, however, much less evident is to prove (5.12). I have not been able to find a simple argument for this, and my proof relies on a fairly explicit computation.

To this end, I write (2.16) as explicitly as possible. Note that

$$\begin{aligned}\left[\frac{\partial^l}{\partial\lambda^l}B_{\lambda,\varepsilon}\right]_{\lambda=0} &= \mathbf{E}\,\frac{\partial^l}{\partial\lambda^l}\left[\exp\left(2\ln\beta\,\frac{d}{dx}\,x\right)\exp\left(\lambda^2\varepsilon\,\frac{d}{dx}\right)\right]_{\lambda=0} T_0\\ &= \sum_{i=0}^{[l/2]}\binom{l}{2i}\mathbf{E}\,\frac{\partial^{l-2i}}{\partial\lambda^{l-2i}}\left[\exp\left(2\ln\beta\,\frac{d}{dx}\,x\right)\right]_{\lambda=0}\left(\varepsilon\,\frac{d}{dx}\right)^i\frac{2i!}{i!}\,T_0\\ &= \sum_{i=0}^{[l/2]}\binom{l}{2i}\mathbf{E}v^{l-2i}\prod_{r=0}^{l-2i-1}\left(2\,\frac{d}{dx}\,x-r\right)\left(\varepsilon\,\frac{d}{dx}\right)^i\frac{2i!}{i!}\,T_0\end{aligned} \tag{5.13}$$

Equation (2.16) for $E=\lambda^2\varepsilon$ can thus be written as

$$\begin{aligned}&\binom{n}{2}4\,\frac{d}{dx}\,x\left[\frac{d}{dx}\,x+\frac{\varepsilon}{4}\,(x+x^{-1})\right]\phi_\varepsilon^{(n-2)}(x)\\ &\quad=\sum_{k=3}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{i=0}^{[l/2]}\binom{l}{2i}\sum_{j=0}^{[(k-l)/2]}\binom{l}{2j}\\ &\quad\times\mathbf{E}v^{l-2i}\mathbf{E}v^{k-l-2j}\,\frac{2i!}{i!}\,\frac{2j!}{j!}\\ &\quad\times\prod_{r=0}^{l-2i-1}\left(2\,\frac{d}{dx}\,x-r\right)\left(\varepsilon\,\frac{d}{dx}\right)^i\prod_{\rho=0}^{k-l-2j-1}\\ &\quad\times\left(-2\,\frac{d}{dx}\,x-\rho\right)\left(\varepsilon\,\frac{d}{dx}\,x^2\right)^j\phi_\varepsilon^{(n-k)}(x)\end{aligned} \tag{5.14}$$

It is convenient to work with the functions $g_\varepsilon^{(n)}((\cdot)\,t)$ and $f_\varepsilon^{(n)}(t)$ introduced above. Both sets of functions are treated completely analogously, and we concentrate on the $g$'s. From (5.14) we derive

$$\begin{aligned}
&\binom{n}{2} 4 \frac{d}{dt}\left[\frac{d}{dt}+\frac{\varepsilon}{2}\cosh t\right] g_\varepsilon^{(n-2)}(t) \\
&\quad = \sum_{k=3}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{i=0}^{[l/2]}\binom{l}{2i}\sum_{j=0}^{[(k-l)/2]}\binom{l}{2j} \\
&\quad \times \mathbf{E}v^{l-2i}\mathbf{E}v^{k-l-2j}\frac{2i!}{i!}\frac{2j!}{j!} \\
&\quad \times \prod_{r=0}^{l-2i-1}\left(2\frac{d}{dt}-r\right)\left(\varepsilon\frac{d}{dt}e^{-t}\right)^{i}\prod_{\rho=0}^{k-l-2j-1} \\
&\quad \times\left(-2\frac{d}{dt}-\rho\right)\left(\varepsilon\frac{d}{dt}e^{t}\right)^{j} g_\varepsilon^{(n-k)}(t)
\end{aligned} \tag{5.15}$$

Equation (2.15) is readily solved for $g_\varepsilon^{(n)}(t)$. The relevant solution is

$$\begin{aligned}
g_\varepsilon^{(n-2)}(t) &= \frac{1}{4\binom{n}{2}} e^{-(\varepsilon/2)\sinh t}\int_{-\infty}^{t} ds\, e^{(\varepsilon/2)\sinh s}\left\{\sum_{k=3}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\right. \\
&\quad \times \sum_{i=0}^{[l/2]}\binom{l}{2i}\sum_{j=0}^{[(k-l)/2]}\binom{l}{2j} \\
&\quad \times \mathbf{E}v^{l-2i}\mathbf{E}v^{k-l-2j}\frac{2i!}{i!}\frac{2j!}{j!}\left[\frac{d}{ds}\right]^{-1}\prod_{r=0}^{l-2i-1}\left(2\frac{d}{ds}-r\right)\left(\varepsilon\frac{d}{ds}e^{-s}\right)^{i} \\
&\quad \times \left.\prod_{\rho=0}^{k-l-2j-1}\left(-2\frac{d}{ds}-\rho\right)\left(\varepsilon\frac{d}{ds}e^{s}\right)^{j} g_\varepsilon^{(n-k)}(s)\right\}
\end{aligned} \tag{5.16}$$

Here I wrote $[d/ds]^{-1}$ instead of an integral in anticipation of the fact that the differential operators following it start with a factor $d/ds$ which will annihilate against it.

Only the term with $k=n$ in (5.16) is completely explicit, since $g_\varepsilon^{(0)}(t)$ is already computed. In the other terms we have to express the $g_\varepsilon^{(n-k)}(t)$ on the right through (5.16) again and iterate this procedure until we are left with a sum of terms involving integrations and differential operators acting on $g_\varepsilon^{(0)}(t)$ only. This is obviously achieved after a finite number of iterations. We estimate the norms of the terms with $k=n$ first. It will then become obvious that these estimates carry over to the other terms as well.

We have to estimate the norms of terms of the form

$$e^{-(\varepsilon/2)\sinh t}\int_{-\infty}^{t} ds\, e^{(\varepsilon/2)\sinh s}\left[\frac{d}{ds}\right]^{-1}\prod_{r=0}^{l-2i-1}\left(2\frac{d}{ds}-r\right)$$
$$\times\left[\varepsilon\frac{d}{ds}(\sinh s+\cosh s)\right]^{i}$$
$$\times\prod_{\rho=0}^{k-l-2j-1}\left(-2\frac{d}{ds}-\rho\right)\left[\varepsilon\frac{d}{ds}(\sinh s+\cosh s)\right]^{j}$$
$$\times e^{-(\varepsilon/2)\sinh s}\int_{-\infty}^{s} ds'\, e^{(\varepsilon/2)\sinh s'} \tag{5.17}$$

The differential operator appearing in this expression can be further expanded. Some constant terms appear in the expansion (i.e., no derivative and no sinh or cosh), but in (5.16) those appear in pairs with opposite signs and thus cancel. All other terms involve a product of powers of the operators $d/ds$, $\varepsilon(d/ds)\sinh s$, and $\varepsilon(d/ds)\cosh s$, with possibly one extra factor of $\varepsilon\sinh s$ or $\varepsilon\cosh s$ to the left. We need therefore to study the action of those operators on $g_\varepsilon^{(0)}(s)$.

We find it convenient to change variables once more to

$$T\equiv\varepsilon\sinh t,\qquad S\equiv\varepsilon\sinh s,\qquad \text{etc.}$$

Note that

$$\frac{d}{dt}=(\varepsilon^2+T^2)^{1/2}\frac{d}{dT} \tag{5.18}$$

We have to estimate integrals of the form

$$I=\int_{-\infty}^{\infty}\frac{dT\, e^{-T/2}}{(\varepsilon^2+T^2)^{1/2}}\int_{-\infty}^{T}\frac{dS\, e^{S/2}}{(\varepsilon^2+S^2)^{1/2}}F(S) \tag{5.19}$$

where $F(S)$ are functions of the form

$$F(S)=\#\prod_{i=1} D^{(\alpha_i)}h(S) \tag{5.20}$$

with

$$h(S)=e^{-S/2}\int_{-\infty}^{S}\frac{dS'\, e^{S'/2}}{(\varepsilon^2+S'^2)^{1/2}}=\int_0^{\infty}\frac{d\tilde{S}\, e^{-\tilde{S}/2}}{[\varepsilon^2+(S-\tilde{S})^2]^{1/2}} \tag{5.21}$$

and the $D^{(\alpha)}$ are either of the operators

$$(\varepsilon^2+S^2)^{1/2}\frac{d}{dS}, \qquad (\varepsilon^2+S^2)^{1/2}\frac{d}{dS}S, \qquad \text{or} \qquad (\varepsilon^2+S^2)^{1/2}\frac{d}{dS}(\varepsilon^2+S^2)^{1/2}$$

and # stands for either 1, $S$, or $(\varepsilon^2+S^2)^{1/2}$.

I first give some estimates on $h(S)$. Notice that $h(S)$ acquires a contribution that is singular in $\varepsilon$ in the integral where $\tilde{S}\sim S$. It will turn out to be useful to single out this singular part and to split $h(S)$ into two pieces,

$$h(S)=h^{(r)}(S)+h^{(s)}(S) \tag{5.22}$$

with

$$h^{(r)}(S)=\int_0^\infty \frac{d\tilde{S}\,e^{-\tilde{S}/2}}{[\varepsilon^2+(S-\tilde{S})^2]^{1/2}}\,\bar{\chi}_S(\tilde{S}) \tag{5.23}$$

$$h^{(s)}(S)=\int_0^\infty \frac{d\tilde{S}\,e^{-\tilde{S}/2}}{[\varepsilon^2+(S-\tilde{S})^2]^{1/2}}\,\chi_S(\tilde{S}) \tag{5.24}$$

The cutoff function $\chi_S(\tilde{S})$ may be chosen as a $C^{(\infty)}$ function with the property that

$$\chi_S(\tilde{S})=\begin{cases}1 & \text{for} \quad |S|<1\\ 1 & \text{for} \quad |S-\tilde{S}|<|S|/2\\ 0 & \text{for} \quad |S-\tilde{S}|>3S/4 \text{ and } |S|\geqslant 1\end{cases} \tag{5.25}$$

Naturally, $\bar{\chi}_S(\tilde{S})\equiv 1-\chi_S(\tilde{S})$. We also define $F^{(r)}(S)$ and $F^{(s)}(S)$ through (5.20) with $h(S)$ replaced by $h^{(r)}(S)$ and $h^{(s)}(S)$, respectively.

Note now that $h^{(r)}$ is an infinitely differentiable function with all its derivatives bounded uniformly in $\varepsilon$. Moreover, $h^{(r)}(S)$ decays like $1/S$, i.e., more precisely, there is a constant $C$ independent of $\varepsilon$ such that

$$|Sh^{(r)}(S)|\leqslant C \tag{5.26}$$

We must show that $h^{(r)}$ retains this property after repeated applications of the operators $D^{(\alpha)}$. This will follow from the following lemma.

**Lemma 3.** Let $f(x)\in C^\infty([-1,1])$ and satisfy $|(1/x)f(x)|<c$. Put

$$d_1(x)=x\frac{d}{dx}f(x) \qquad \text{and} \qquad d_2(x)=x\frac{d}{dx}\frac{1}{x}f(x)$$

Then, $d_1(x)$ and $d_2(x)$ are in $C^\infty([-1, 1])$ and there are constants $c_1$ and $c_2$ such that

$$\left|\frac{1}{x} d_{1,2}(x)\right| \leqslant c_{1,2}$$

for all $x \in [-1, 1]$.

The proof of this lemma employs Taylor's formula with remainder and is left as an exercise.

The application to our case is evident. Considering $h^{(r)}$ as a function of the variable $x = 1/S$ and remembering that it vanishes for $|S| < 1$ by the definition of $\bar{\chi}_S(\tilde{S})$, we see that it satisfies the assumptions of Lemma 3. Moreover, $x\, d/dx = S\, d/dS$, and the distinction between $S$ and $(\varepsilon^2 + S^2)^{1/2}$ is insignificant. We obtain therefore that

$$|F^{(r)}(S)| \leqslant C \tag{5.27}$$

for some constant $C$ independent of $\varepsilon$. The corresponding contribution to the integral $I$ is thus bounded by a constant times $|\ln |\varepsilon| \,|^2$, as desired.

I turn now to $h^{(s)}$. It is easily seen to satisfy the bound

$$|h^{(s)}(S)| \leqslant Ce^{-S/5} \,|\ln |\varepsilon| \,|\, \chi[S > -1] \tag{5.28}$$

Now notice that

$$\frac{d}{dS} \frac{1}{[\varepsilon^2 + (S - \tilde{S})^2]^{1/2}} = -\frac{d}{d\tilde{S}} \frac{1}{[\varepsilon^2 + (S - \tilde{S})^2]^{1/2}} \tag{5.29}$$

by means of which we get, performing an integration by parts, that

$$\frac{d}{dS} h^{(s)} = \int_0^\infty \frac{d\tilde{S}\, e^{-\tilde{S}/2}}{[\varepsilon^2 + (S - \tilde{S})^2]^{1/2}} \left[ -\frac{1}{2} \chi_S(\tilde{S}) + \frac{d}{d\tilde{S}} \chi_S(\tilde{S}) + \frac{d}{dS} \chi_S(\tilde{S}) \right] \tag{5.30}$$

This satisfies virtually the same bounds as $h^{(s)}$ itself. Using the same strategy for the operators repeatedly, we see that $F^{(s)}$ can be expressed as a sum of terms, all of which are bounded by terms like

$$(\varepsilon^2 + S^2)^{1/2}\, Ce^{-S/5} \,|\ln |\varepsilon| \,|\, \chi[s > -1]$$

with $k \geqslant 0$ and constants $C$ independent of $\varepsilon$. The important point to notice is that there is at least one factor of $(\varepsilon^2 + S^2)^{1/2}$ and that the function has exponential decay in $S$. Inserted in (5.19), the exponential decay will guarantee the convergence of the $T$ integral, and the $(\varepsilon^2 + S^2)^{1/2}$ factor removes the $\varepsilon$ singularity in the $S$ integral. Thus, these two integrals produce only an $|\ln|\varepsilon||$ factor that combines with the $\ln \varepsilon$ in the bound on $F^{(s)}$ to give again a constant times $|\ln|\varepsilon||^2$ bound for this contribution to $I$. This is what I wanted to show.

This concludes the proof for the terms with $k=n$ in (5.16). The general case is treated in the same way. The structure of the terms to be estimated is

$$e^{-\sinh}\int \varepsilon^{\sinh} De^{-\sinh}\int e^{\sinh} De^{-\sinh}\cdots De^{-\sinh}\int e^{\sinh}$$

with the $D$'s representing differential operators of the same type as before. Thus, the same structure as in the previous case is repeated several times, and the proof is obtained by induction in a fairly obvious way, using in each step the same estimates as just done. I will not present the details here. The final result is that

$$\| g_\varepsilon^{(n)}(x)\|_1 \leqslant c_n\, |\ln \varepsilon|^2$$

The same procedure applied to the functions $f_\varepsilon^{(n)}(x)$ shows that they satisfy the same bounds, which completes the proof of the theorem.

It is natural to ask whether the approximate invariant measures we compute in perturbation theory converge to some measure as $\varepsilon$ approaches zero. The answer to this question is in fact easily derived as a corollary of the theorem. We have the following result.

**Corollary I.** Let

$$d\nu_\varepsilon^{(n)}(x) \equiv \frac{\sum^n \phi_\varepsilon^{(k)}(x)\, \lambda^k/k!}{\|\sum^n \phi_\varepsilon^{(k)}(x)\, \lambda^k/k!\|_1}\, dx$$

Then, for all $\lambda$, as $\varepsilon \to 0$ the measure $d\nu_\varepsilon^{(n)}(x)$ converges weakly to the measure $\frac{1}{2}[\delta(x)+(1/x^2)\,\delta(1/x)]\,dx$.

I leave the proof of this corollary to the reader.

I now turn to the computation of the Lyapunov exponent and of the density of states. Starting from (2.7), one can express $\gamma(E)$ in terms of the $t$ variables as

$$\begin{aligned}\gamma(E) &= \frac{1}{\|\phi_{\lambda,E}(x)\|_1}\left[\int_{-\infty}^{0} dx\, \ln(-x)\, \phi_{\lambda,E}(x) + \int_0^\infty dx\, \ln x\, \phi_{\lambda,E}(x)\right] + \mathbf{E}\ln\beta \\ &= \frac{\int_{-\infty}^{\infty} t(g_{\lambda,\varepsilon}(t)-f_{\lambda,\varepsilon}(t))\, dt}{\|g_{\lambda,\varepsilon}(t)\|_1 + \|f_{\lambda,\varepsilon}(t)\|_1} + \mathbf{E}\ln\beta \qquad (5.31)\end{aligned}$$

Similarly, from (2.8) the density of states is given by

$$N(E)=\frac{\|f_{\lambda,\varepsilon}(t)\|_1}{\|g_{\lambda,\varepsilon}(t)\|_1+\|f_{\lambda,\varepsilon}(t)\|_1} \tag{5.32}$$

I will (formally) expand the Lyapunov exponent and the density of states in powers of $\lambda$, i.e., for $E=\varepsilon\lambda^2$, I let

$$\begin{aligned}\gamma(E)&=\sum_{n=0}^{\infty}\lambda^n\gamma^{(n)}(\varepsilon)\\ N(E)&=\frac{1}{2}+\sum_{n=2}^{\infty}\lambda^n N^{(n)}(\varepsilon)\end{aligned} \tag{5.33}$$

I will prove the following theorem:

**Theorem 2.** The Lyapunov exponent vanishes to all orders in perturbation theory as $E\to 0$, while the density of states approaches the value 1/2. More precisely, for all $n$ there exist constants $c_n$ and $d_n$ such that, up to corrections vanishing faster with $\varepsilon$,

$$\gamma^{(n)}(\varepsilon)=\frac{c_n}{|\ln|\varepsilon||} \tag{5.34}$$

and for $n\geqslant 2$,

$$N^{(n)}(\varepsilon)=\frac{d_n}{|\ln|\varepsilon||^2}\operatorname{sign}\varepsilon \tag{5.35}$$

As a consequence of (5.35), the differentiated density of states

$$n(E)\equiv\frac{1}{\lambda^2}\frac{d}{d\varepsilon}N(\varepsilon)$$

diverges at the band center like $\varepsilon^{-1}|\ln\varepsilon|^{-3}$, to all orders of perturbation theory. This extends the earlier results of Dyson[5] and Markos.[9]

*Proof.* Let us turn first to the density of states. Since $f$ and $g$ are positive functions, we can write (5.32) as

$$N(E)=\frac{1}{2}+\frac{\int_{-\infty}^{\infty}[g_{\lambda,\varepsilon}(t)-f_{\lambda,\varepsilon}(t)]\,dt}{\|g_{\lambda,\varepsilon}(t)\|_1+\|f_{\lambda,\varepsilon}(t)\|_1} \tag{5.36}$$

The integral over the difference between $f$ and $g$ can now be expanded in powers of $\lambda$, and the coefficients can be expressed using (2.14). [Note that the integral over $f(t)$ equals the integral over $f(-t)$.] That is, we use that

$$g_\varepsilon^{(n)}(t)-f_\varepsilon^{(n)}(-t)=\sum_{k=2}^{n}\binom{n}{k}\sum_{i=0}^{[k/2]}\binom{k}{2i}\frac{2i!}{i!}\mathbf{E}v^{k-2i} \times\prod_{r=0}^{k-2i-1}\left(2\frac{d}{dt}-r\right)\left(\varepsilon\frac{d}{dt}e^t\right)^i f_\varepsilon^{(n-k)}(-t) \tag{5.37}$$

Notice that the right-hand side is a total derivative and that therefore the integral is just given by boundary terms at plus and minus infinity. Since all the $f_\varepsilon^{(n)}(t)$ decay like $1/(\varepsilon\cosh t)$ at infinity [this is checked easily using l'Hôpital's rule for $f_\varepsilon^{(0)}(t)$, and then proven by induction in a similar manner as in the proof of Theorem 1], a moment's reflection shows that these boundary terms are in fact of the order of a constant, i.e., independent of $\varepsilon$. Therefore, to any order in $\lambda$,

$$N(E)=\frac{1}{2}+\frac{a}{|\ln\varepsilon|^2} \tag{5.38}$$

Differentiating this formula with respect to $E$, we obtain that the differentiated density of states $n(E)$ diverges to any order in $\lambda$ like

$$n(E)=\frac{b}{|\varepsilon\ln^3\varepsilon|} \tag{5.39}$$

We now turn to the Lyapunov exponent.

We expand the $\phi_{\lambda,E}(x)$ in the numerator of (5.31) in powers of $\lambda$ and use (2.12). The term proportional to $\lambda^n$ is then given by

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{[k/2]}\binom{k}{2i}\frac{2i!}{i!}\mathbf{E}v^{k-2i} \times\int_{-\infty}^{\infty}dt\,t\prod_{r=0}^{k-2i-1}\left(2\frac{d}{dt}-r\right)\left(\varepsilon\frac{d}{dt}e^t\right)^i f_\varepsilon^{(n-k)}(-t) \tag{5.40}$$

In the last integral we may perform a partial integration to obtain[3]

$$\begin{aligned}&-\int_{-\infty}^{\infty}dt\left[\frac{d}{dt}\right]^{-1}\prod_{r=0}^{k-2i-1}\left(2\frac{d}{dt}-r\right)\left(\varepsilon\frac{d}{dt}e^t\right)^i f_\varepsilon^{(n-k)}(-t)\\&\quad=-\int_{-\infty}^{\infty}dt\prod_{r=0}^{k-2i-1}\left(2\frac{d}{dt}-r\right)\varepsilon e^t\left(\varepsilon\frac{d}{dt}e^t\right)^{i-1} f_\varepsilon^{(n-k)}(-t)\end{aligned} \tag{5.41}$$

[3] Apart from a boundary term that is nonsingular in $\varepsilon$ and may thus be ignored.

Now notice that just as in the proof of Theorem 1, we can show that all integrals of the form

$$\int_{-\infty}^{\infty} dt \left(\frac{d}{dt}\right)^{j} \varepsilon e^{t} \left(\varepsilon \frac{d}{dt} e^{t}\right)^{i-1} f_{\varepsilon}^{(n-k)}(-t)$$

with $i>0$ and $j>0$ or $i=0$ and $j>1$ are bounded by const $\cdot \ln |\varepsilon|$. Therefore,

$$\begin{aligned}
&\int_{0}^{\infty} dx \ln x(1-T_0)\, \phi_{\lambda,E}(x) \\
&\quad = -\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathbf{E} v^k 2 \int_{-\infty}^{\infty} dt \prod_{r=1}^{k-1} (-r)\, f_{\varepsilon}^{(n-k)}(-t) + o(\ln \varepsilon) \\
&\quad = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathbf{E} v^k (k-1)!\, (-1)^k \, \|\phi_{\varepsilon}^{(n-k)}(x)\|_1 + o(\ln \varepsilon) \\
&\quad = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \|\phi_{\varepsilon}^{(n)}(x)\|_1 \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k} \mathbf{E} v^k + o(\ln \varepsilon) \\
&\quad = -\|\phi_{\lambda,E}(x)\|_1 \, \mathbf{E} \ln \beta + o(\ln \varepsilon)
\end{aligned} \tag{5.42}$$

This shows that indeed

$$\gamma(E) = 0 + O(\ln^{-1} |\varepsilon|)$$

to all orders of perturbation theory, as claimed.

Before closing this section, I point out that the point $E=0$ can be analyzed in great detail for all $\lambda$. This has been exploited partially in ref. 13. Note that the Schrödinger equation takes the form

$$\beta_{n+1}\psi(n+1) + \beta_n \psi(n-1) = 0$$

so that the even and odd sublattices decouple. Two linearly independent solutions are obtained by choosing initial conditions

$$\psi^{(+)}(0) = 0, \qquad \psi^{(+)}(1) = 1$$

and

$$\psi^{(-)}(0) = 1, \qquad \psi^{(-)}(1) = 0$$

Then $\psi^{(+)}$ vanishes on the even sublattice and $\psi^{(-)}$ on the odd one, while

$$\psi^{(+)}(2n+1) = \prod_{k=1}^{n} \frac{\beta(2k)}{\beta(2k+1)} \tag{5.43}$$

and

$$\psi^{(-)}(2n) = \prod_{k=1}^{n} \frac{\beta(2k-1)}{\beta(2k)} \tag{5.44}$$

Thus

$$\ln |\psi^{(+)}(2n+1)| = \sum_{k=1}^{n} [\ln |\beta(2k)| - \ln |\beta(2k+1)|] \tag{5.45}$$

$$\ln |\psi^{(-)}(2n)| = \sum_{k=1}^{n} [\ln |\beta(2k-1)| - \ln |\beta(2k)|] \tag{5.46}$$

With

$$\gamma(k) \equiv \ln |\beta(2k)| - \ln |\beta(2k+1) \tag{5.47}$$

we have

$$\ln |\psi^{(+)}(2n+1)| = \sum_{k=1}^{n} \gamma(k) \tag{5.48}$$

and

$$\ln |\psi^{(-)}(2n+1)| = -\sum_{k=1}^{n} \gamma(k) + \beta(2n+1) - \beta(0) \tag{5.49}$$

The $\gamma(k)$ are i.i.d. random variables with mean zero variance $\sigma$ of order $\lambda$. Thus, by the central limit theorem,

$$\Phi_n \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \gamma(k)$$

converges to a Gaussian random variable $\Phi$ with mean zero and variance $\sigma$. Of course,

$$\frac{1}{n} \sum_{k=1}^{n} \gamma(k) \to 0, \quad \text{a.s.}$$

and the Lyapunov exponent is thus zero, as predicted by perturbation theory. [I leave it to the reader to verify the vanishing of the Lyapunov exponent directly from the definition (1.5).]

The two solutions thus have the asymptotic behavior

$$|\psi^{(+)}(2n+1)| \sim \exp(+n^{1/2}\Phi)$$

and

$$|\psi^{(-)}(2n)| \sim \exp(-n^{1/2}\Phi)$$

where $\Phi$ does depend on the realization of the disorder. Typically, therefore, as long as $\lambda \neq 0$, the solutions grow faster than any power either in the forward or backward direction, and therefore cannot be generalized eigenstates, which exhibit at most algebraic growth. Since for $E \neq 0$ we even have a positive Lyapunov exponent, we expect that the spectrum of $H$ is still pure point, with all eigenfunctions localized, although with divergent localization length at zero energy. It would be very interesting to prove this conjecture rigorously.

## 6. MIXED DISORDER

It is a natural question to ask what happens in a model where both diagonal and off-diagonal disorder is present, i.e., for a Hamilonian of the form

$$H_{\text{mix}} = -\Delta + \delta V + \lambda J \tag{6.1}$$

with $V$ and $J$ defined as in Section 1, and the random variables $V_i$ independent of the $v_i$. The corresponding Schrödinger equation is then

$$\beta_{n+1}\psi_E(n+1) - \beta_n\psi_E(n-1) + (\delta v_n - E)\,\psi_E(n) = 0 \tag{6.2}$$

With $x_n$ defined as in (2.2), we get from (6.2) the recursion relation

$$x_{n+1} = \frac{E - \delta v_n - x_n^{-1}}{\beta_{n+1}^2} \tag{6.3}$$

and for the density of the corresponding invariant measure we derive the equation

$$\begin{aligned}\phi_{\lambda,\delta}(x) &= \mathbf{E}_V\mathbf{E}_v\left[\frac{\beta^2}{(E-\delta V-\beta^2 x)^2}\,\phi_{\lambda,\delta}\left(\frac{1}{E-\delta V-\beta^2 x}\right)\right]\\ &= \mathbf{E}_V\mathbf{E}_v\left[\exp\left(2\ln\beta\,\frac{d}{dx}\,x\right)\exp\left(\delta V\frac{d}{dx}\right)\right] T_E\phi_{\lambda,\delta}(x)\end{aligned} \tag{6.4}$$

For $E \neq 0$, nothing very interesting happens, and I consider only the case $E = 0$. I want to take both $\lambda$ and $\delta$ small of the same order and put $\delta = \rho\lambda$. One may than expand $\phi_{\lambda,\delta}(x)$ in powers of $\lambda$ and obtain for the expansion coefficients the systems of equations

$$\binom{n}{2} A_{0,\rho}\phi_\rho^{(n-2)}(x) = -\sum_{k=3}^{n}\binom{n}{k}\left[\frac{\partial^k}{\partial\lambda^k}B_{\lambda,\rho}\right]_{\lambda=0}\phi_\rho^{(n-k)}(x) \tag{6.5}$$

where now

$$A_{0,\rho} = 8\,\frac{d}{dx}\left[x\,\frac{d}{dx}\,x + \frac{\rho^2}{8}(1+x^4)\,\frac{d}{dx} + \frac{\rho^2}{2}\,x^3\right] \tag{6.6}$$

To lowest order this gives

$$\phi_\rho^{(0)}(x) = \frac{1}{[x^2 + \rho/8(1+x^4)]^{1/2}} \tag{6.7}$$

which is normalizable for finite $\rho$, and whose norm diverges like $|\ln \rho|$ as $\rho$ goes to zero. One might repeat all the analysis of the previous section for this case now, and exhibit the singular behavior of the density of states and the Lyapunov exponent in all orders of perturbation theory. I will not, however, go any further with this. My main point is just the observation that any amount of added diagonal disorder will remove the singular behavior at the band center, and will in particular restore exponential localization. This fact was already noted in ref. 13.

To conclude, I summarize the results as follows. Using the invariant measure approach, it is possible to construct for one-dimensional Schrödinger operators a perturbation expansion with finite coefficients to all orders. From those it is possible to investigate in detail the singular behavior of the density of states and the Lyapunov exponents near special energies, in particular the band center. The problem that is still open is to reconstruct from the formal perturbation expansions the actual solutions. The least one would like to prove is that the perturbation theory is asymptotic; the best one might hope for is to show Borel summability.

## NOTE ADDED IN PROOF

Recently, M. Campanino and A. Klein (U. C. Irvine, preprint) proved the asymptotic nature of the improved perturbation expansion in the random potential model of ref. 2.

## REFERENCES


1. P. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators* (Birkhäuser, Boston, 1985).
2. A. Bovier and A. Klein, *J. Stat. Phys.* **51**:501 (1988).
3. M. Campanino and J. F. Perez, *Commun. Math. Phys.* (1989), to appear.
4. B. Derrida and E. Gardner, *J. Phys.* (Paris) **45**:1283 (1984).
5. F. Dyson, *Phys. Rev.* **92**:1331 (1953).
6. H. Fürstenberg, *Trans. Am. Math. Soc.* **108**:377 (1963).
7. M. Kappus and F. Wegner, *Z. Phys. B* **45**:15 (1981).
8. P. Markos, *J. Phys. C* **21**:2147 (1988).
9. P. Markos, *Z. Phys. B* **73**:17 (1988).
10. D. Ruelle, *IHES Publ.* **50**:275 (1979).
11. E. Roman and C. Wiecko, *Z. Phys. B* **62**:163 (1986); **69**:81 (1987).
12. B. Simon and M. Taylor, *Commun. Math. Phys.* **101**:1 (1985).
13. G. Theodorou and M. H. Cohen, *Phys. Rev. B* **13**:4597 (1976).